```
> # This example is example 2 in Chapter 5 of the book Modelling population dynamics:  model
      formulation, fitting and assessment  using state−space methods
  # Press enter to activate each line of code (or use !!! button above to active all the code)
> restart;
> with(LinearAlgebra) :
> Dmat := proc(kappa, pars)
  local DD1, i, j;
  description "This proceedure form the derivative matrix given a vector of exhaustive summary
      terms, kappa, and a vector of parameters, pars";
  with(LinearAlgebra) :
  DD1 := Matrix(1 ..Dimension(pars), 1 ..Dimension(kappa)) :
  for i from 1 to Dimension(pars)  do
      for j from 1 to Dimension(kappa) do
          DD1[i, j] := diff(kappa[j], pars[i])
      end do
   end do;
  DD1;
   end proc:
> Estpars := proc(DD1, pars)
  local r, d, alphapre, alpha, PDE, FF, i, ans;
  description "Finds the estimable set of parameters for derivative matrix DD1 and vector of
      parameters pars";
  with(LinearAlgebra) :
  r := Rank(DD1);
  d := Dimension(pars) - r :
  alphapre := NullSpace(Transpose(DD1)) :
  α := Matrix(d, Dimension(pars)) : PDE := Vector(d) :
  FF := f(seq(pars[i], i = 1 ..Dimension(pars))) :
  for i from 1 to d do
        α[i, 1 ..Dimension(pars)] := alphapre[i] :
        PDE[i] := add(diff(FF, pars[j])·α[i, j], j = 1 ..Dimension(pars)) :
   end do:
   ans := pdsolve({seq(PDE[i] = 0, i = 1 ..d)});
  end proc:
> Estpars2 := proc(DD1, pars)
  local r, d, alphapre, alpha, PDE, FF, i, ans;
  description "Finds the estimable set of parameters for derivative matrix DD1 and vector of
      parameters pars. Also returns the vector alpha and set of PDEs";
  with(LinearAlgebra) :
  r := Rank(DD1);
  d := Dimension(pars) - r :
  alphapre := NullSpace(Transpose(DD1)) :
  α := Matrix(d, Dimension(pars)) : PDE := Vector(d) :
  FF := f(seq(pars[i], i = 1 ..Dimension(pars))) :
  for i from 1 to d do
        α[i, 1 ..Dimension(pars)] := alphapre[i] :
        PDE[i] := add(diff(FF, pars[j])·α[i, j], j = 1 ..Dimension(pars)) :
   end do:
   ans := pdsolve({seq(PDE[i] = 0, i = 1 ..d)});
   ⟨{ans}; {alpha}; {PDE}⟩;
```

```
end proc:
> Formnum := proc(D1)
 local results, j, numpars, D1rand :

    description  "This procedure finds the rank and alpha for the hybrid-symbolic-numeric
    method";
results := Matrix(5, 2) :
for j from 1 to 5 do
   numpars := seq(indets(D1)[i] = evalf(rand()/1000000000000), i = 1 ..nops(indets(D1))) :
   D1rand := eval(D1, {numpars});
   results[j, 1] := Rank( D1rand);
  results[j, 2] := NullSpace(Transpose(D1rand)) :
 end do:
 results :
 end proc:
> # Forming the vector kappa, which is an exhaustive summary consisting of capture probabilties:
> kappa := ⟨ϕ[1]·p[2], ϕ[1]·(1-p[2])·ϕ[2]·p[3], ϕ[1]·(1-p[2])·ϕ[2]·(1-p[3])·ϕ[3]
      ·p[4], ϕ[2]·p[3], ϕ[2]·(1-p[3])·ϕ[3]·p[4], ϕ[3]·p[4]⟩;
```

$$\kappa := \begin{bmatrix} \phi_1 p_2 \\ \phi_1 (1-p_2) \phi_2 p_3 \\ \phi_1 (1-p_2) \phi_2 (1-p_3) \phi_3 p_4 \\ \phi_2 p_3 \\ \phi_2 (1-p_3) \phi_3 p_4 \\ \phi_3 p_4 \end{bmatrix} \quad (1)$$

```
> #The vector of parameters:
> pars := ⟨ϕ[1]|ϕ[2]|ϕ[3]|p[2]|p[3]|p[4]⟩;
```

$$pars := \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & p_2 & p_3 & p_4 \end{bmatrix} \quad (2)$$

```
> #Using the prodcedure Dmat to find the derivative matrix
> D1 := Dmat(kappa, pars);
```

**(3)**

$$D1 := \begin{bmatrix} p_2 & (1-p_2)\,\phi_2 p_3 & (1-p_2)\,\phi_2\,(1-p_3)\,\phi_3 p_4 & 0 & 0 & 0 \\ 0 & \phi_1\,(1-p_2)\,p_3 & \phi_1\,(1-p_2)\,(1-p_3)\,\phi_3 p_4 & p_3 & (1-p_3)\,\phi_3 p_4 & 0 \\ 0 & 0 & \phi_1\,(1-p_2)\,\phi_2\,(1-p_3)\,p_4 & 0 & \phi_2\,(1-p_3)\,p_4 & p_4 \\ \phi_1 & -\phi_1\,\phi_2 p_3 & -\phi_1\,\phi_2\,(1-p_3)\,\phi_3 p_4 & 0 & 0 & 0 \\ 0 & \phi_1\,(1-p_2)\,\phi_2 & -\phi_1\,(1-p_2)\,\phi_2\,\phi_3 p_4 & \phi_2 & -\phi_2\,\phi_3 p_4 & 0 \\ 0 & 0 & \phi_1\,(1-p_2)\,\phi_2\,(1-p_3)\,\phi_3 & 0 & \phi_2\,(1-p_3)\,\phi_3 & \phi_3 \end{bmatrix} \quad \textbf{(3)}$$

> #Finding the rank of the derivative matrix and the deficiency. A deficiency of greater than zero means the model is parameter redundant.

> $r := Rank(D1);\ d := Dimension(pars) - r;$

$$r := 5$$
$$d := 1 \quad \textbf{(4)}$$

> #Finding the estimable parameter combinations:

> $Estpars(D1, pars);$

$$\left\{ f\left(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4\right) = \_F1\left(\phi_1, \phi_2, p_2, p_3, \phi_3\, p_4\right) \right\} \quad \textbf{(5)}$$

> #Finding the estimable parameter combinations, and also displaying alpha and the PDEs:

> $Estpars2(D1, pars);$

$$\begin{bmatrix} \left\{ \left\{ f\left(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4\right) = \_F1\left(\phi_1, \phi_2, p_2, p_3, \phi_3\, p_4\right) \right\} \right\} \\ \left\{ \begin{bmatrix} 0 & 0 & -\dfrac{\phi_3}{p_4} & 0 & 0 & 1 \end{bmatrix} \right\} \\ \left\{ \left[ -\dfrac{\left(\dfrac{\partial}{\partial \phi_3} f\left(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4\right)\right)\phi_3}{p_4} + \dfrac{\partial}{\partial p_4} f\left(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4\right) \right] \right\} \end{bmatrix} \quad \textbf{(6)}$$

>

> # The hybrid-formal numerical method:

> $Formnum(D1)$

$$
\left[\begin{array}{c}
5\ \left\{\begin{bmatrix} 6.01407028563251\ 10^{-13} \\ -7.43515288406911\ 10^{-12} \\ -0.999646079371163 \\ -2.97418332880497\ 10^{-13} \\ 3.31321559468765\ 10^{-12} \\ 0.0266029321290483 \end{bmatrix}\right\} \\
5\ \left\{\begin{bmatrix} 9.07604450759213\ 10^{-17} \\ -7.36538265867933\ 10^{-13} \\ 0.883982239398018 \\ 3.27694867488509\ 10^{-18} \\ 9.49393673055948\ 10^{-13} \\ -0.467520481293457 \end{bmatrix}\right\} \\
5\ \left\{\begin{bmatrix} 8.20316155359517\ 10^{-12} \\ -1.07949462516365\ 10^{-10} \\ -0.433650034933370 \\ -4.14199859665352\ 10^{-12} \\ 5.64593893215752\ 10^{-11} \\ 0.901081376570556 \end{bmatrix}\right\} \\
5\ \left\{\begin{bmatrix} -2.79038262015414\ 10^{-11} \\ 1.40607416976417\ 10^{-10} \\ -0.537582069728446 \\ 2.42663254260799\ 10^{-11} \\ -9.64959920993744\ 10^{-11} \\ 0.843211431555858 \end{bmatrix}\right\} \\
5\ \left\{\begin{bmatrix} -1.34759673474735\ 10^{-13} \\ 8.12704141425436\ 10^{-11} \\ 0.928360587608024 \\ 1.43392686120758\ 10^{-12} \\ -5.34806672512640\ 10^{-11} \\ \ldots \end{bmatrix}\right\}
\end{array}\right. \qquad \textbf{(7)}
$$

> # This also shows the model has rank 5. The alpha entries close to zero indicate that $\phi_1$, $\phi_2$, $\rho_2$ and $p_3$ can be estimated.

>

> # An alternative exhaustive summary that based on the means is:

> kappa := ⟨N[1]·φ[1]·p[2], N[1]·φ[1]·(1-p[2])·φ[2]·p[3], N[1]·φ[1]·(1-p[2])·φ[2]·(1-p[3])·φ[3]·p[4], N[1]·(1 − φ[1]·p[2] − φ[1]·(1-p[2])·φ[2]·p[3] − φ[1]·(1-p[2])·φ[2]·(1-p[3])·φ[3]·p[4]), N[2]·φ[2]·p[3], N[2]·φ[2]·(1-p[3])·φ[3]·p[4], N[2]·(1 − φ[2]·p[3] − φ[2]·(1-p[3])·φ[3]·p[4]), N[3]·φ[3]·p[4], N[3]·(1 − φ[3]·p[4])⟩;

$$\kappa := \left[\begin{array}{c} N_1 \phi_1 p_2 \\ N_1 \phi_1 (1-p_2) \phi_2 p_3 \\ N_1 \phi_1 (1-p_2) \phi_2 (1-p_3) \phi_3 p_4 \\ N_1 \left(1 - \phi_1 p_2 - \phi_1 (1-p_2) \phi_2 p_3 - \phi_1 (1-p_2) \phi_2 (1-p_3) \phi_3 p_4\right) \\ N_2 \phi_2 p_3 \\ N_2 \phi_2 (1-p_3) \phi_3 p_4 \\ N_2 \left(1 - \phi_2 p_3 - \phi_2 (1-p_3) \phi_3 p_4\right) \\ N_3 \phi_3 p_4 \\ N_3 (1 - \phi_3 p_4) \end{array}\right] \quad \textbf{(8)}$$

> #The vector of parameters:

> pars := ⟨φ[1]|φ[2]|φ[3]|p[2]|p[3]|p[4]⟩;

$$pars := \left[\begin{array}{cccccc} \phi_1 & \phi_2 & \phi_3 & p_2 & p_3 & p_4 \end{array}\right] \quad \textbf{(9)}$$

> #Using the prodcedure Dmat to find the derivative matrix

> D1 := Dmat(kappa, pars);

$$D1 := \left[\left[N_1 p_2, N_1 (1-p_2) \phi_2 p_3, N_1 (1-p_2) \phi_2 (1-p_3) \phi_3 p_4, N_1 \left(-p_2 - (1-p_2) \phi_2 p_3 - (1-p_2) \phi_2 (1-p_3) \phi_3 p_4\right), 0, 0, 0, 0, 0\right],\right.$$
$$\left[0, N_1 \phi_1 (1-p_2) p_3, N_1 \phi_1 (1-p_2) (1-p_3) \phi_3 p_4, N_1 \left(-\phi_1 (1-p_2) p_3 - \phi_1 (1-p_2) (1-p_3) \phi_3 p_4\right), N_2 p_3, N_2 (1-p_3) \phi_3 p_4, N_2 \left(-p_3 - (1-p_3) \phi_3 p_4\right), 0, 0\right],$$
$$\left[0, 0, N_1 \phi_1 (1-p_2) \phi_2 (1-p_3) p_4, -N_1 \phi_1 (1-p_2) \phi_2 (1-p_3) p_4, 0, N_2 \phi_2 (1-p_3) p_4, -N_2 \phi_2 (1-p_3) p_4, N_3 p_4, -N_3 p_4\right],$$
$$\left[N_1 \phi_1, -N_1 \phi_1 \phi_2 p_3, -N_1 \phi_1 \phi_2 (1-p_3) \phi_3 p_4, N_1 \left(-\phi_1 + \phi_1 \phi_2 p_3 + \phi_1 \phi_2 (1-p_3) \phi_3 p_4\right), 0, 0, 0, 0, 0\right], \quad \textbf{(10)}$$

$$\left[0, N_1 \phi_1 (1-p_2) \phi_2, -N_1 \phi_1 (1-p_2) \phi_2 \phi_3 p_4, N_1 \left(-\phi_1 (1-p_2) \phi_2 + \phi_1 (1-p_2) \phi_2 \phi_3 p_4\right), N_2 \phi_2, -N_2 \phi_2 \phi_3 p_4, N_2 \left(-\phi_2 + \phi_2 \phi_3 p_4\right), 0, 0\right],$$
$$\left[0, 0, N_1 \phi_1 (1-p_2) \phi_2 (1-p_3) \phi_3, -N_1 \phi_1 (1-p_2) \phi_2 (1-p_3) \phi_3, 0, N_2 \phi_2 (1-p_3) \phi_3, -N_2 \phi_2 (1-p_3) \phi_3, N_3 \phi_3, -N_3 \phi_3\right]\right]$$

> #Finding the rank of the derivative matrix and the deficiency. A deficiency of greater than zero means the model is parameter redundant.

> $r := Rank(D1); d := Dimension(pars) - r;$

$$r := 5$$
$$d := 1 \quad \textbf{(11)}$$

> #Finding the estimable parameter combinations:

> $Estpars(D1, pars);$

$$\left\{ f(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4) = \_F1(\phi_1, \phi_2, p_2, p_3, \phi_3 p_4) \right\} \quad \textbf{(12)}$$

> #Finding the estimable parameter combinations, and also displaying alpha and the PDEs:

> $Estpars2(D1, pars);$

$$\left[\begin{array}{c} \left\{ \left\{ f(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4) = \_F1(\phi_1, \phi_2, p_2, p_3, \phi_3 p_4) \right\} \right\} \\ \left\{ \left[\begin{array}{cccccc} 0 & 0 & -\frac{\phi_3}{p_4} & 0 & 0 & 1 \end{array}\right] \right\} \\ \left\{ \left[ -\frac{\left(\frac{\partial}{\partial \phi_3} f(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4)\right) \phi_3}{p_4} + \frac{\partial}{\partial p_4} f(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4) \right] \right\} \end{array}\right] \quad \textbf{(13)}$$

>

> # A simpler exhaustive summary given in Cole et al (2010, Mathematical Biosciences) is:

> # forming the vector kappa, which is an exhaustive summary consisting of capture probabilties

> kappa := ⟨ϕ[1]·p[2], ϕ[1]·(1-p[2]), ϕ[2]·p[3], ϕ[2]·(1-p[3]), ϕ[3]·p[4]⟩;

$$\kappa := \left[\begin{array}{c} \phi_1 p_2 \\ \phi_1 (1-p_2) \\ \phi_2 p_3 \\ \phi_2 (1-p_3) \\ \phi_3 p_4 \end{array}\right] \quad \textbf{(14)}$$

> #The vector of parameters:

> $pars := \langle \phi[1] | \phi[2] | \phi[3] | p[2] | p[3] | p[4] \rangle;$

$$pars := \left[\begin{array}{cccccc} \phi_1 & \phi_2 & \phi_3 & p_2 & p_3 & p_4 \end{array}\right] \quad \textbf{(15)}$$

> #Using the prodcedure Dmat to find the derivative matrix

> $D1 := Dmat(\text{kappa}, pars);$

$$D1 := \begin{bmatrix} p_2 & 1-p_2 & 0 & 0 & 0 \\ 0 & 0 & p_3 & 1-p_3 & 0 \\ 0 & 0 & 0 & 0 & p_4 \\ \phi_1 & -\phi_1 & 0 & 0 & 0 \\ 0 & 0 & \phi_2 & -\phi_2 & 0 \\ 0 & 0 & 0 & 0 & \phi_3 \end{bmatrix} \quad (16)$$

> #Finding the rank of the derivative matrix and the deficiency. A deficiency of greater than zero means the model is parameter redundant.

> $r := Rank(D1); d := Dimension(pars) - r;$

$$r := 5$$
$$d := 1 \quad (17)$$

> #Finding the estimable parameter combinations:

> $Estpars(D1, pars);$

$$\left\{ f(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4) = \_F1(\phi_1, \phi_2, p_2, p_3, \phi_3 p_4) \right\} \quad (18)$$

> #Finding the estimable parameter combinations, and also displaying alpha and the PDEs:

> $Estpars2(D1, pars);$

$$\begin{bmatrix} \left\{ \left\{ f(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4) = \_F1(\phi_1, \phi_2, p_2, p_3, \phi_3 p_4) \right\} \right\} \\ \left\{ \begin{bmatrix} 0 & 0 & -\frac{\phi_3}{p_4} & 0 & 0 & 1 \end{bmatrix} \right\} \\ \left\{ \begin{bmatrix} -\frac{\left( \frac{\partial}{\partial \phi_3} f(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4) \right) \phi_3}{p_4} + \frac{\partial}{\partial p_4} f(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4) \end{bmatrix} \right\} \end{bmatrix} \quad (19)$$

>

> # The hybrid-formal numerical method:

> $Formnum(D1)$

$$
\left[\begin{array}{c}
5 \quad \left\{\left[\left[\begin{array}{c}
6.01407028563251\ 10^{-13} \\
-7.43515288406911\ 10^{-12} \\
-0.999646079371163 \\
-2.97418332880497\ 10^{-13} \\
3.31321559468765\ 10^{-12} \\
0.0266029321290483
\end{array}\right]\right]\right\} \\
5 \quad \left\{\left[\left[\begin{array}{c}
9.07604450759213\ 10^{-17} \\
-7.36538265867933\ 10^{-13} \\
0.883982239398018 \\
3.27694867488509\ 10^{-18} \\
9.49393673055948\ 10^{-13} \\
-0.467520481293457
\end{array}\right]\right]\right\} \\
5 \quad \left\{\left[\left[\begin{array}{c}
8.20316155359517\ 10^{-12} \\
-1.07949462516365\ 10^{-10} \\
-0.433650034933370 \\
-4.14199859665352\ 10^{-12} \\
5.64593893215752\ 10^{-11} \\
0.901081376570556
\end{array}\right]\right]\right\} \\
5 \quad \left\{\left[\left[\begin{array}{c}
-2.79038262015414\ 10^{-11} \\
1.40607416976417\ 10^{-10} \\
-0.537582069728446 \\
2.42663254260799\ 10^{-11} \\
-9.64959920993744\ 10^{-11} \\
0.843211431555858
\end{array}\right]\right]\right\} \\
5 \quad \left\{\left[\left[\begin{array}{c}
-1.34759673474735\ 10^{-13} \\
8.12704141425436\ 10^{-11} \\
0.928360587608024 \\
1.43392686120758\ 10^{-12} \\
-5.34806672512640\ 10^{-11} \\
\text{[illegible]}
\end{array}\right]\right]\right\}
\end{array}\right] \qquad \textbf{(20)}
$$