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> # This example is example 1 in Chapter 5 of the book Modelling population dynamics:  model
      formulation, fitting and assessment using state-space methods
  # Press enter to activate each line of code (or use !!! button above to active all the code)
> restart;
> with(LinearAlgebra) :
> Dmat := proc(kappa, pars)
  local DD1, i, j;
  description "This proceedure form the derivative matrix given a vector of exhaustive summary
      terms, kappa, and a vector of parameters, pars";
  with(LinearAlgebra) :
  DD1 := Matrix(1 ..Dimension(pars), 1 ..Dimension(kappa)) :
  for i from 1 to Dimension(pars)  do
      for j from 1 to Dimension(kappa) do
          DD1[i, j] := diff(kappa[j], pars[i])
      end do
   end do;
  DD1;
   end proc:
> PLUR := proc(D1)
   local pp, ll, u1, r1, DetU;
   description   "This procedure finds a PLUR or turing factorisation of the matrix D1." :
  (pp, ll, u1, r1) := LUDecomposition( D1, output = ['P','L','U1','R'] ) :
  DetU := Determinant(u1);
   ⟨DetU = 0, {P = pp, L = ll, U = u1, R = r1}⟩ :
   end proc:
> Formnum := proc(D1)
   local results, j, numpars, D1rand :

      description   "This procedure finds the rank and alpha for the hybrid-symbolic-numeric
      method";
  results := Matrix(5, 2) :
  for j from 1 to 5 do
     numpars := seq(indets(D1)[i] = evalf(rand( ) / 1000000000000), i = 1 ..nops(indets(D1))) :
     D1rand := eval(D1, {numpars});
     results[j, 1] := Rank( D1rand);
    results[j, 2] := NullSpace(Transpose(D1rand)) :
   end do:
   results :
   end proc:
> # forming the vector kappa which consists of the mean terms
```

> kappa := $\langle n\cdot p_1\cdot(1-\psi\cdot p_2), n\cdot(1-p_1)\cdot p_2, n\cdot p_1\cdot\psi\cdot p_2, n-n\cdot p_1\cdot(1-\psi\cdot p_2)-n\cdot(1-p_1)\cdot p_2 -n\cdot p_1\cdot\psi\cdot p_2\rangle$;

$$\kappa := \begin{bmatrix} n\,p_1\left(1-\psi\,p_2\right) \\ n\left(1-p_1\right)p_2 \\ n\,p_1\,\psi\,p_2 \\ n-n\,p_1\left(1-\psi\,p_2\right)-n\left(1-p_1\right)p_2-n\,p_1\,\psi\,p_2 \end{bmatrix}$$

```
> #The vector of parameters:
> pars := <ψ, p[1], p[2]>;
```

$$pars := \begin{bmatrix} \psi \\ p_1 \\ p_2 \end{bmatrix}$$

```
> #Using the prodcedure Dmat to find the derivative matrix:
> D1 := simplify(Dmat(kappa, pars));
```

$$D1 := \begin{bmatrix} -n\,p_1\,p_2 & 0 & n\,p_1\,p_2 & 0 \\ -n\left(-1+\psi\,p_2\right) & -n\,p_2 & n\,\psi\,p_2 & -n+n\,p_2 \\ -n\,p_1\,\psi & -n\left(-1+p_1\right) & n\,p_1\,\psi & n\left(-1+p_1\right) \end{bmatrix}$$

```
> #Finding the rank of the derivative matrix and the deficiency. A deficiency of greater than zero
      means the model is parameter redundant.
> r := Rank(D1); d := Dimension(pars) - r;
```

$$r := 3$$
$$d := 0$$

```
> # Using a PLUR decomposition to find any points where the model is not full rank.
> PLURresults := PLUR(D1);
```

$$PLURresults := \left[\left[-n^3\,p_1\,p_2\left(-1+p_1\right)=0\right],\right.$$

$$\left[\left[\left\{L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1+\psi\,p_2}{p_1\,p_2} & 1 & 0 \\ \frac{\psi}{p_2} & \frac{-1+p_1}{p_2} & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, U\right.\right.\right.$$

$$= \left[\begin{array}{ccc} -n\,p_1\,p_2 & 0 & n\,p_1\,p_2 \\ 0 & -n\,p_2 & n \\ 0 & 0 & -\frac{(-1+p_1)\,n}{p_2} \end{array}\right]$$

> #The model will be parameter redundant for any solutions of $-n^3\, p_1\, p_2\, (-1+p_1) = 0$ for which P, L, R and U are defined.
> # So the model is parameter redundant if n=0 and if $p_1 = 1$.
> #The model may be parameter redundant if p_1 or $p_2 = 0$, but L, U and R would be undefined in these cases so we need to check them seperatley.

> $Rank(eval(D1, p_1 = 0))$

$$2$$

> $Rank(eval(D1, p_2 = 0))$

$$2$$

> # As these are rank 2 rather than 3 the model is parameter redundant if p_1 or $p_2 = 0$

>

> # The hybrid-formal numerical method:

> $Formnum(D1)$

$$\left[\begin{array}{cc} 3 & \{\,\} \\ 3 & \{\,\} \\ 3 & \{\,\} \\ 3 & \{\,\} \\ 3 & \{\,\} \end{array}\right]$$

> # This also shows the model has rank 3 (as the model is full rank alpha={})

>

> # A simpler exhaustive summary is (as long as n is not zero).

> kappa := $\langle p_1 \cdot (1 - \psi \cdot p_2), (1 - p_1) \cdot p_2, p_1 \cdot \psi \cdot p_2 \rangle$;

$$\kappa := \left[\begin{array}{c} p_1\,(1 - \psi\, p_2) \\ (1 - p_1)\, p_2 \\ p_1\, \psi\, p_2 \end{array}\right]$$

> #The vector of parameters:

> $pars := \langle \psi, p_1, p_2 \rangle$;

$$pars := \begin{bmatrix} \psi \\ p_1 \\ p_2 \end{bmatrix}$$

> #Using the prodcedure Dmat to find the derivative matrix:

> $D1 := Dmat(\text{kappa}, pars);$

$$D1 := \begin{bmatrix} -p_1 p_2 & 0 & p_1 p_2 \\ 1 - \psi p_2 & -p_2 & \psi p_2 \\ -p_1 \psi & 1 - p_1 & p_1 \psi \end{bmatrix}$$

> #Finding the rank of the derivative matrix and the deficiency. Note that we again find this model has rank 3.

> $r := Rank(D1); d := Dimension(pars) - r;$

$$r := 3$$

$$d := 0$$

> # A PLUR decomposition can be used with the new exhaustive summary and we get the same results, except for the result that the model is parameter redundant if n=0, as this is now an implicit assumption of the new exhaustive summary.

> $PLURresults := PLUR(D1);$

$$PLURresults := \left[\left[-p_1 p_2 \left(-1 + p_1\right) = 0\right], \left[\left\{L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1 + \psi p_2}{p_1 p_2} & 1 & 0 \\ \frac{\psi}{p_2} & \frac{-1 + p_1}{p_2} & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, U = \begin{bmatrix} -p_1 p_2 & 0 & p_1 p_2 \\ 0 & -p_2 & 1 \\ 0 & 0 & -\frac{-1 + p_1}{p_2} \end{bmatrix}\right\}\right]\right]$$

> #The model will be parameter redundant for any solutions of $-p_1 p_2 \left(-1 + p_1\right) = 0$ for which P,

L, R and U are defined. So model is parameter redundant if n = 0 and if $p_1 = 1$. Model may be parameter redundant if p_1 or $p_2 = 0$, but L and R would be undefined in these cases so we need to check them seperatley.

```
> Rank(eval(D1, p[1] = 0))
```

$$2$$

```
> Rank(eval(D1, p[2] = 0))
```

$$2$$

> # As these are rank 2 rather than 3 the model is parameter redundant if p_1 or $p_2 = 0$

>

> # The hybrid-formal numerical method also give a rank of 3 again:

```
> Formnum(D1)
```

$$\begin{bmatrix} 3 & \{\} \\ 3 & \{\} \\ 3 & \{\} \\ 3 & \{\} \\ 3 & \{\} \end{bmatrix}$$

> # This also shows the model has rank 3 (as model is full rank alpha={})

>